Liquidity Risk Estimation in Conditional Volatility Models

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Motivation

In risk management,

liquidity often associated with simple **transaction costs**

Explains why *liquidity adjusted* risk measures are of the form

market risk measure + **liquidity term**

where

market risk measure : obtained from historical market prices

liquidity term : obtained from **bid-ask spreads** data

Motivation

But Market risk measures obtained from **historical market prices** already include a liquidity component

Why ? Liquidity has a direct impact on historical price variations

Our objective in this paper is to **extract** this liquidity component from global risk measures computed from historical prices:

global risk measure = market risk measure + **liquidity term**

where

global risk measure : obtained from **historical market prices**

market risk measure : obtained from **historical market prices**

Main contribution

Our **liquidity risk measure:**

- is defined as a **intrinsec characteristic** of a given asset and allows simple liquidity rankings

- takes into account the dynamic properties of prices (**time varying** market risk)

- can be defined for **different *conditional* risk measures** (VaR, Expected Shortfall, …)

- can be computed when **only historical market prices** are available

Agenda

1. Global risk and global risk-parameter

2. Additive decomposition of global risk

3. Inference

4. Empirical Applications

1. Global risk and Global risk-parameter

1st STEP Volatility modeling

GARCH(1,1) governs the returns process

$$\varepsilon_t = \sigma_t(\theta_0)\eta_t, \quad \eta_t \; i.i.d. \;\; E\eta_t^2 = 1$$
$$\sigma_t^2(\theta_0) = \omega_0 + a_0\varepsilon_{t-1}^2 + b_0\sigma_{t-1}^2$$

- $\theta_0 = (\omega_0, a_0, b_0)'$ is a ***volatility***-parameter
- captures the volatility persistence in asset returns
- this model can be easily generalized to more complex conditional volatility models

1. Global risk and Global risk-parameter

2nd STEP Global risk measure

The *conditional* VaR of ε_t at level α is

$$P_{t-1}\left\lfloor \varepsilon_t < -VaR_t^G(\alpha) \right\rfloor = \alpha$$

With the previous specification of ε_t, the ***global risk***

$$VaR_t^G(\alpha) = -\sigma_t(\theta_0)\, F_\eta^{-1}(\alpha)$$

depends on

- the dynamics of the GARCH process through $\sigma_t(\theta_0)$
- the (constant) lower tail of the innovation process

1. Global risk and Global risk-parameter

3rd STEP Global risk-parameter (Francq, Zakoian (2012))

A0 (scale stability) *There exists a function H such that for any θ, for any K > 0, and any sequence (x_i)*

$$K\sigma(x_1, x_2, \ldots ; \theta) = \sigma(x_1, x_2, \ldots ; \theta^*); \quad \text{where} \quad \theta^* = H(\theta; K)$$

We can then concentrate in a single *global risk-parameter* $\theta_{0,\alpha}$ the 2 dimensions of risk

$$\theta_{0,\alpha}^{G} = H\left(\theta_0, -F_\eta^{-1}(\alpha)\right)$$

and obtain the global risk as

$$VaR_t^G(\alpha) = \sigma_t\left(\theta_{0,\alpha}^G\right)$$

1. Global risk and Global risk-parameter

3rd STEP Global risk-parameter (Francq, Zakoian (2012))

In our GARCH(1,1) example …

$$\theta_{0,\alpha}^{G} = \left(K^2\omega_0, K^2 a_0, b_0\right)$$
$$K = -F_{\eta}^{-1}(\alpha)$$

… but ***A0*** is also satisfied for more complex GARCH specification (power-transformed asymmetric GARCH model)

2. Additive decomposition of global risk

We need the following assumption to identify both global and market risks from returns

A1 (Identification assumption) *For an infinitely liquid asset, the innovations of the GARCH(1,1) process are Gaussian*

We define the *market risk-parameter* as

$$\theta_{0,\alpha}^{M} = H\left(\theta_0, -\Phi^{-1}(\alpha)\right)$$

and the corresponding market risk is

$$\boxed{VaR_t^M(\alpha) = \sigma_t\left(\theta_{0,\alpha}^{M}\right)}$$

2. Additive decomposition of global risk

Interpretation of A1 (Identification assumption)

Usual way used to include liquidity shocks (*see Duffie, Pan (1997), An Overview of value at risk*)

$$\varepsilon_t = \sigma_t(\theta_0)\eta_t + Jump, \quad \eta_t \; i.i.d. \; Gaussian$$

2. Additive decomposition of global risk

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$$\varepsilon_t = \sigma_t(\theta_0)\eta_t + Jump, \quad \eta_t \; i.i.d.\; Gaussian$$

One step further (*see Meddahi, Mykland (2012), Fat Tails or Many Small Jumps ?)*

$$\varepsilon_t = \sigma_t(\theta_0)\chi_t + Jump, \quad \chi_t \; i.i.d.\; Student$$

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Our approach

$$\varepsilon_t = \sigma_t(\theta_0)\chi_t = \sigma_t(\theta_0)\eta_t + \sigma_t(\theta_0)(-\eta_t + \chi_t)$$

2. Additive decomposition of global risk

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Our approach

« LIQUIDITY »

$$\varepsilon_t = \sigma_t(\theta_0)\chi_t = \sigma_t(\theta_0)\eta_t + \boxed{\sigma_t(\theta_0)(-\eta_t + \chi_t)}$$

2. Additive decomposition of global risk

A2 (Consistency assumption) $0 > \Phi^{-1}(\alpha) > F_{\eta}^{-1}(\alpha)$ *for a sufficient small* α

Definition The *liquidity risk-parameter* is (*for a small enough*)

$$\theta_{0,\alpha}^{L} = H\left(\theta_0, -F_{\eta}^{-1}(\alpha) + \Phi^{-1}(\alpha)\right)$$

and the corresponding liquidity risk is

$$\boxed{VaR_t^L(\alpha) = \sigma_t\left(\theta_{0,\alpha}^L\right)}$$

Proposition Under A0-A2,

$$VaR_t^G(\alpha) = VaR_t^M(\alpha) + VaR_t^L(\alpha)$$

3. Inference

Two-step approach

1st STEP

$\hat{\theta}_n$: Gaussian QML estimator of θ_0 (does not require to know the distribution of η_t)

2nd STEP

$\xi_{n,\alpha}$: nonparametric estimator of the innovation quantile function ξ_α, obtained from

$$\hat{\eta}_t = \varepsilon_t / \sigma_t\left(\hat{\theta}_n\right)$$

Final STEP

$$\hat{\theta}^G_{n,\alpha} = H\left(\hat{\theta}_n, -\xi_{n,\alpha}\right) \quad \hat{\theta}^M_{n,\alpha} = H\left(\hat{\theta}_n, -\Phi^{-1}\right) \quad \hat{\theta}^G_{n,\alpha} = H\left(\hat{\theta}_n, -\xi_{n,\alpha} + \Phi^{-1}\right)$$

3. Inference

Asymptotic distribution follows from the joint distribution of

$$\left(\hat{\theta}, \xi_{n,\alpha}\right)$$

In the general case of power-transformed asymmetric GARCH model

$$\varepsilon_t = \sigma_t\left(\theta_0\right)\eta_t, \quad \eta_t \; i.i.d. \;\; E\eta_t^2 = 1$$

$$\sigma_t^\delta\left(\theta_0\right) = \omega_0 + \sum_{i=1}^{q} \alpha_{0i+}\left(\varepsilon_{t-i}^{+}\right)^\delta + \alpha_{0i-}\left(-\varepsilon_{t-i}^{-}\right)^\delta + \sum_{j=1}^{p} \beta_{0j}\sigma_{t-j}^\delta$$

$$x^+ = \max(x,0), \quad x^- = \min(x,0)$$

3. Inference

We use the following technical assumptions (same as those required for the Gaussian QML)

D: $\theta_0 \in \Theta$ and Θ is compact; $\gamma < 0$; $\forall\theta \in \Theta$, $\sum_{j=1}^{p} \beta_j < 1$ and $\omega > \underline{\omega}$ for some $\underline{\omega} > 0$; if $P(\eta_t \in \Gamma) = 1$ for a set Γ, then Γ has a cardinal $|\Gamma| > 2$; $P[\eta_t > 0] \in (0, 1)$; if $p > 0$, $\mathcal{B}_{\theta_0}(z)$ has no common root with $\mathcal{A}_{\theta_0+}(z)$ and $\mathcal{A}_{\theta_0-}(z)$. Moreover $\mathcal{A}_{\theta_0+}(1) + \mathcal{A}_{\theta_0-}(1) \neq 0$ and $\alpha_{0q,+} + \alpha_{0q,-} + \beta_{0p} \neq 0$.

3. Inference

Let $D_t(\theta) = \dfrac{1}{\sigma_t(\theta)} \dfrac{\partial \sigma_t(\theta)}{\partial \theta} = \dfrac{1}{\delta} \dfrac{1}{\sigma_t^\delta(\theta)} \dfrac{\partial \sigma_t^\delta(\theta)}{\partial \theta}$

If η_1 admits a continuous and strictly positive density *f* in a neigborhood of ξ_α, we have

$$\begin{pmatrix} \sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \\ \sqrt{n}\left(\xi_{n,\alpha} - \xi_\alpha\right) \end{pmatrix} \to N\left(0, \Sigma_\alpha\right) \quad \Sigma_\alpha = \begin{pmatrix} \dfrac{\kappa_4 - 1}{4} J^{-1} & \lambda_\alpha \delta \overline{\theta}_0 \\ \lambda_\alpha \delta \overline{\theta}_0' & \varsigma_\alpha \end{pmatrix}$$

where

$$\lambda_\alpha = \xi_\alpha \frac{\kappa_4 - 1}{4} + \frac{p_\alpha}{2 f(\xi_\alpha)}, \quad p_\alpha = E\left(\eta_1^2 1_{\{\eta_1 < \xi_\alpha\}}\right) - \alpha$$

$$\varsigma_\alpha = \xi_\alpha^2 \frac{\kappa_4 - 1}{4} + \frac{\xi_\alpha p_\alpha}{f(\xi_\alpha)} + \frac{\alpha(1 - \alpha)}{f^2(\xi_\alpha)}$$

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1. does not depend on θ_0

3. Inference

Let $D_t(\theta) = \frac{1}{\sigma_t(\theta)} \frac{\partial \sigma_t(\theta)}{\partial \theta} = \frac{1}{\delta} \frac{1}{\sigma_t^\delta(\theta)} \frac{\partial \sigma_t^\delta(\theta)}{\partial \theta}$

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2. usual *i.i.d.* case

positive or negative

3. Inference

For $X \in \{G, M, L\}$

$$\sqrt{n}\left(\hat{\theta}_{n,\alpha}^{X} - \theta_{0,\alpha}^{X}\right) \rightarrow N\left(0, \Delta_{\alpha}^{X}\right)$$

$$\Delta_{\alpha}^{L} = \frac{\kappa_4 - 1}{4} A_L \left(J^{-1} - \delta^2 \frac{\xi_{\alpha}^2}{\left\{-\xi_{\alpha} + \Phi^{-1}(\alpha)\right\}^2} \bar{\theta}_0 \bar{\theta}_0' \right) A_L$$

$$+ \delta^2 \left\{-\xi_{\alpha} + \Phi^{-1}(\alpha)\right\}^{2(\delta-1)} \left\{ 2\lambda_{\alpha} \Phi^{-1}(\alpha) + \frac{\alpha(1-\alpha)}{f^2(\xi_{\alpha})} \right\} \bar{\theta}_0 \bar{\theta}_0'$$

3. Inference

For $X \in \{G, M, L\}$

$$\sqrt{n}\left(\hat{\theta}^{X}_{n,\alpha} - \theta^{X}_{0,\alpha}\right) \to N\left(0, \Delta^{X}_{\alpha}\right)$$

$$\Delta^{L}_{\alpha} = \frac{\kappa_4 - 1}{4} A_L \left(J^{-1} - \delta^2 \frac{\xi^2_\alpha}{\left\{-\xi_\alpha + \Phi^{-1}(\alpha)\right\}^2} \bar{\theta}_0 \bar{\theta}_0' \right) A_L$$

$$+ \delta^2 \left\{-\xi_\alpha + \Phi^{-1}(\alpha)\right\}^{2(\delta-1)} \left\{ 2\lambda_\alpha \Phi^{-1}(\alpha) + \frac{\alpha(1-\alpha)}{\boxed{f^2(\xi_\alpha)}} \right\} \bar{\theta}_0 \bar{\theta}_0'$$

Only the density at ξ_α must be estimated

3. Inference

For $X \in \{G, M, L\}$

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Only the density at ξ_α must be estimated

$$Va\hat{R}^{L}_{t}(\alpha) = \sigma_t\left(\hat{\theta}^{L}_{n,\alpha}\right)$$

4. Empirical Applications – *stock market data*

50 constituants of the Eurostoxx 50 index, as of September 27, 2012 – ***Blue chips « liquid » stocks***

3131 Daily log returns, from September 27, 2000 to September 26, 2012

Using a ***GARCH(1,1) specification*** *of the conditional volatility, we estimate for all stocks*

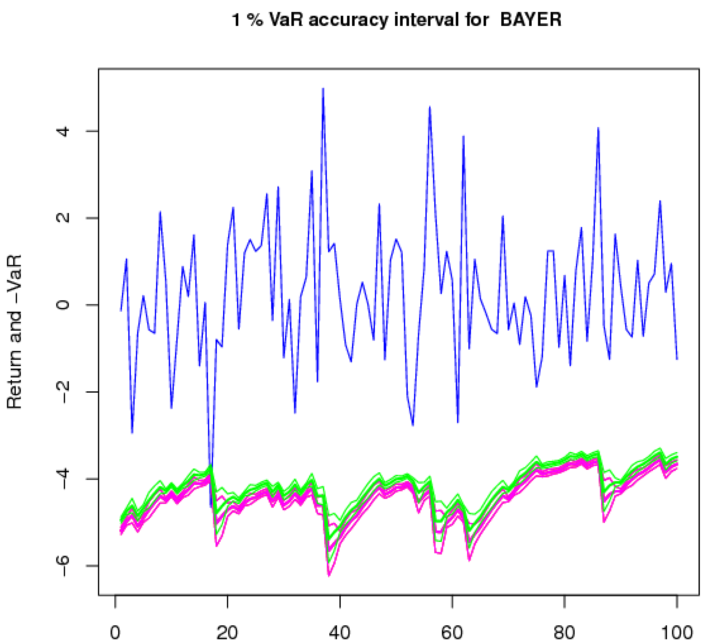
$$VaR_t^G(0.01) \quad VaR_t^M(0.01) \quad VaR_t^L(0.01)$$

and the corresponding risk parameters

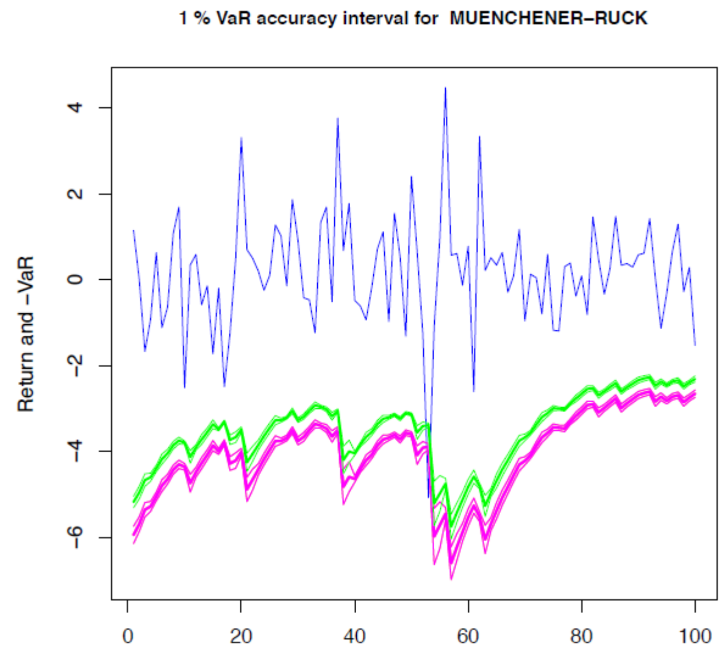
$$\hat{\theta}_{n,0.01}^G \quad \hat{\theta}_{n,0.01}^M \quad \hat{\theta}_{n,0.01}^L$$

4. Empirical Applications – *stock market data*

2 examples



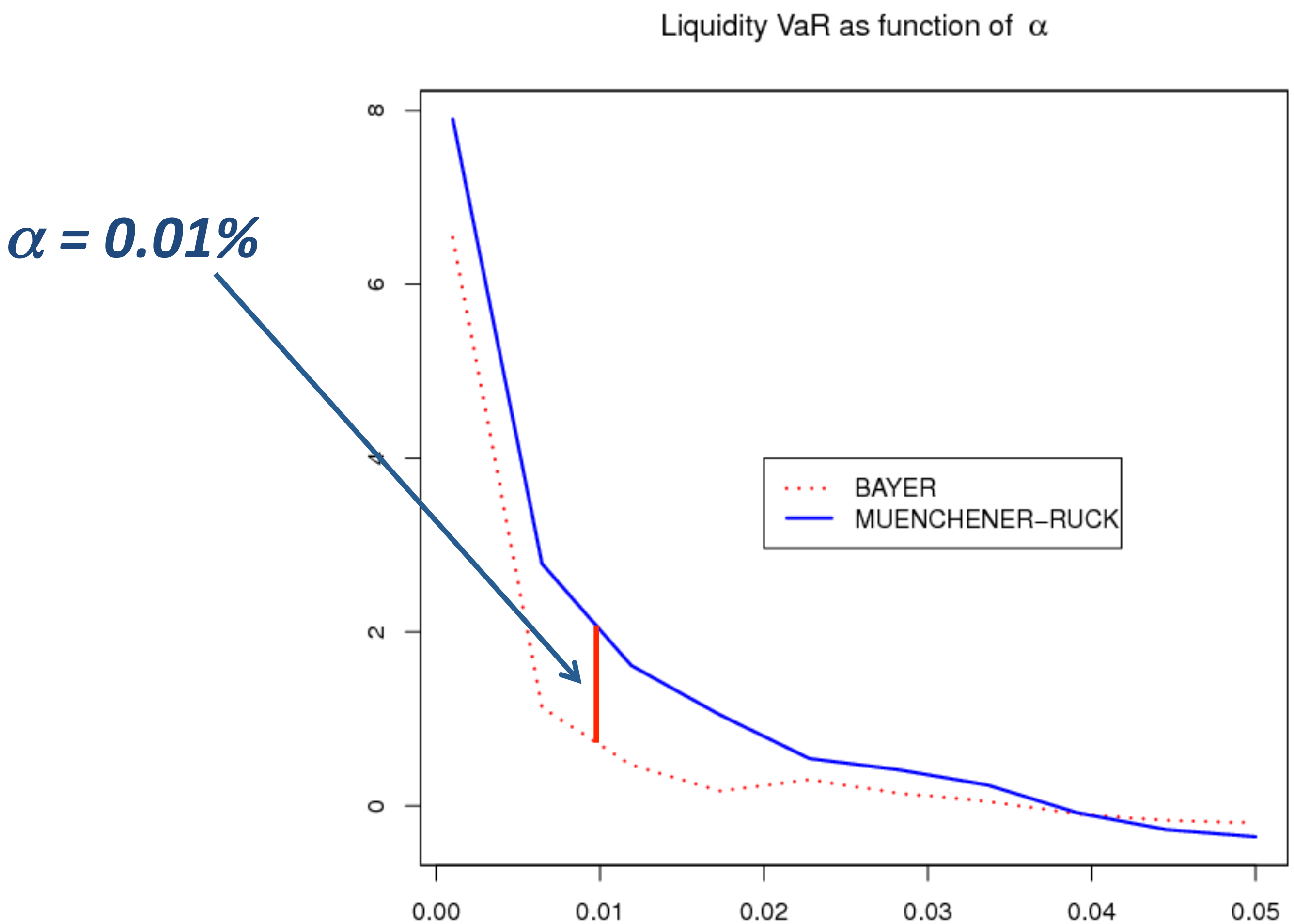
VaR parameter: (ω,α,β): Global= (0.47,0.5,0.9), Market= (0.43,0.45,0.9)



VaR parameter: (ω,α,β): Global= (0.38,0.86,0.87), Market= (0.28,0.65,0.87)

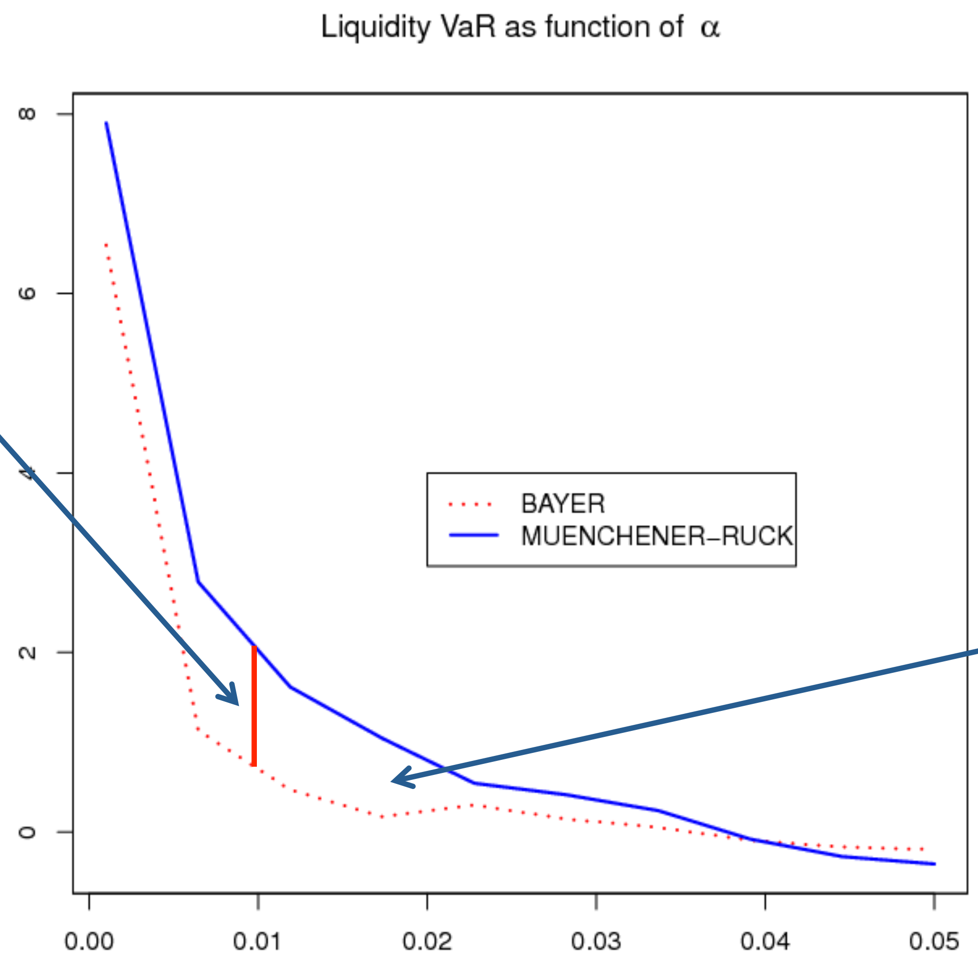
4. Empirical Applications – *stock market data*

For different levels of α



4. Empirical Applications – *stock market data*

For different levels of α

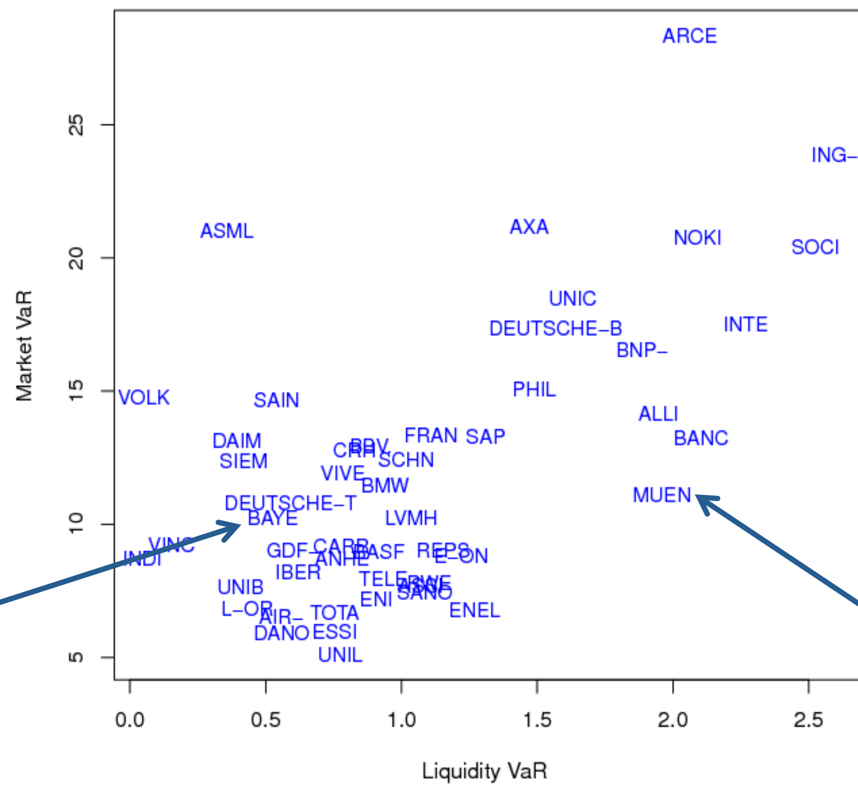


α = 0.01%

Low liquidity risk when α > 0.01%

4. Empirical Applications – *stock market data*

A representation of the investment universe in terms of liquidity risk

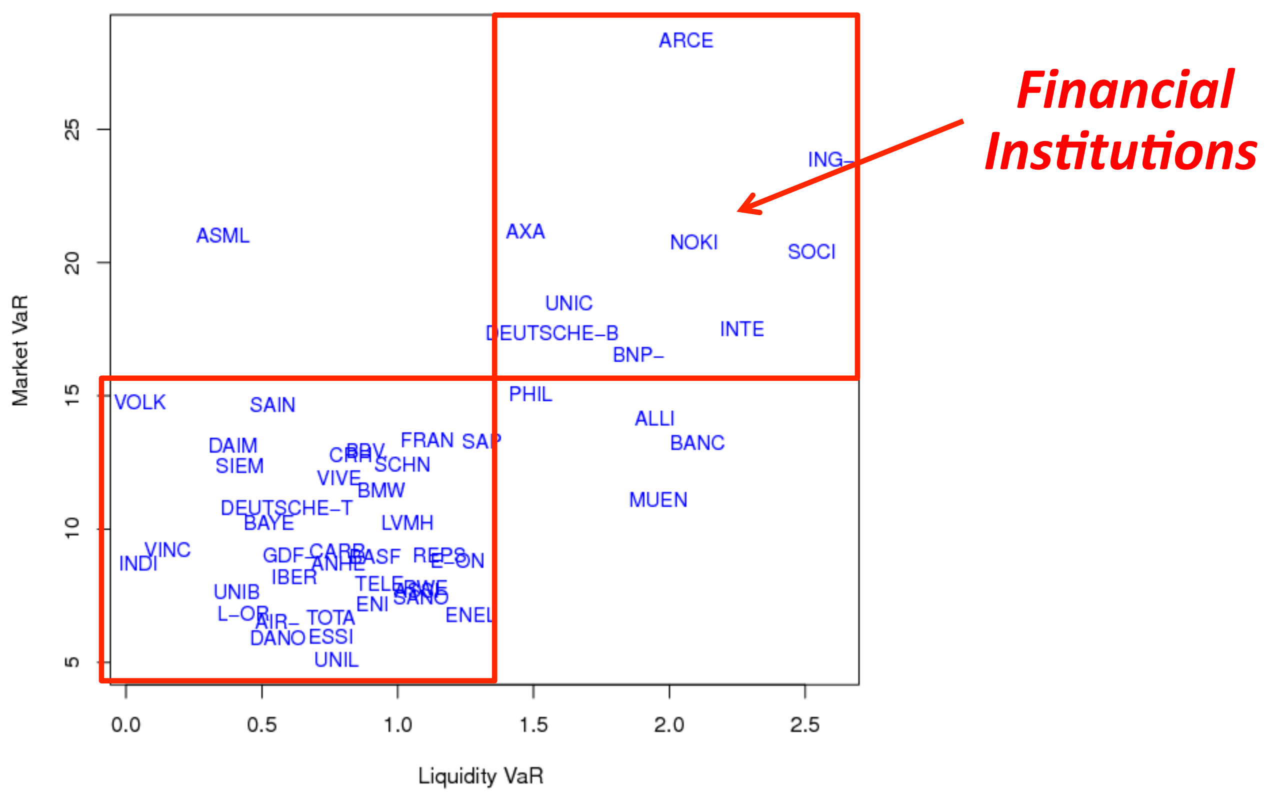


Bayer

Muenchener

4. Empirical Applications – *stock market data*

A representation of the investment universe in terms of liquidity risk



4. Empirical Applications – *hedge funds data*

14 Lyxor hedge funds strategy indices, and a composite index – ***Big differences in terms of liquidity risk exposure***

560 weekly returns, from April 16, 2002 to December 31, 2012

Using a ***GARCH(1,1) specification*** *of the conditional volatility, we estimate for all stocks*

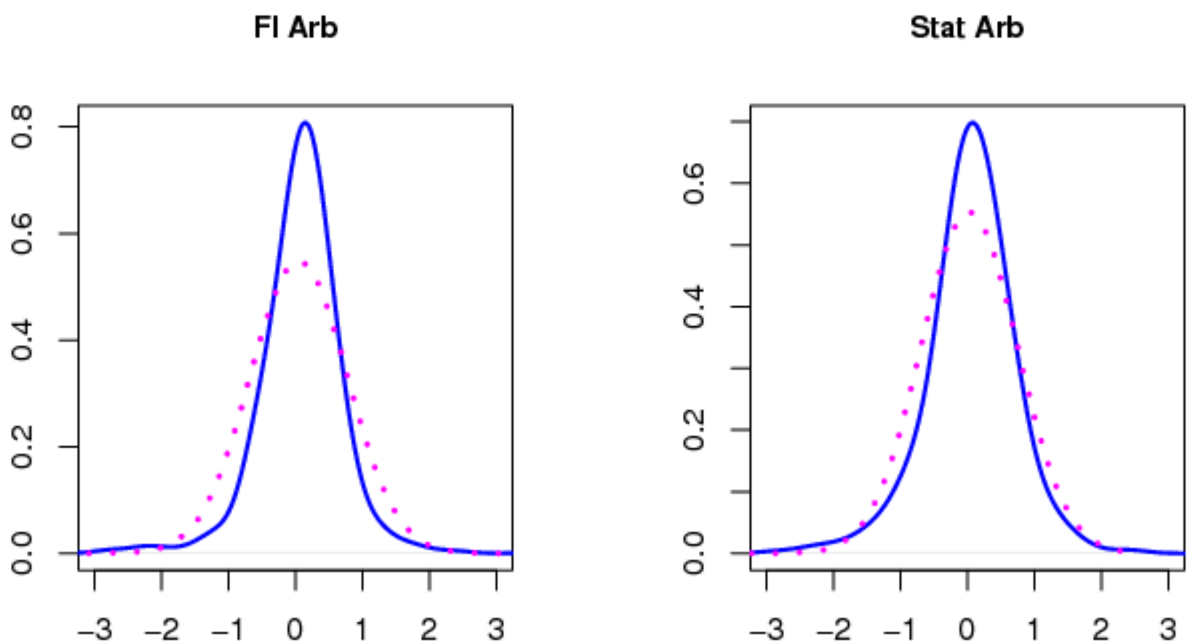
$$VaR_t^G(0.01) \quad VaR_t^M(0.01) \quad VaR_t^L(0.01)$$

and the corresponding risk parameters

$$\hat{\theta}_{n,0.01}^G \quad \hat{\theta}_{n,0.01}^M \quad \hat{\theta}_{n,0.01}^L$$

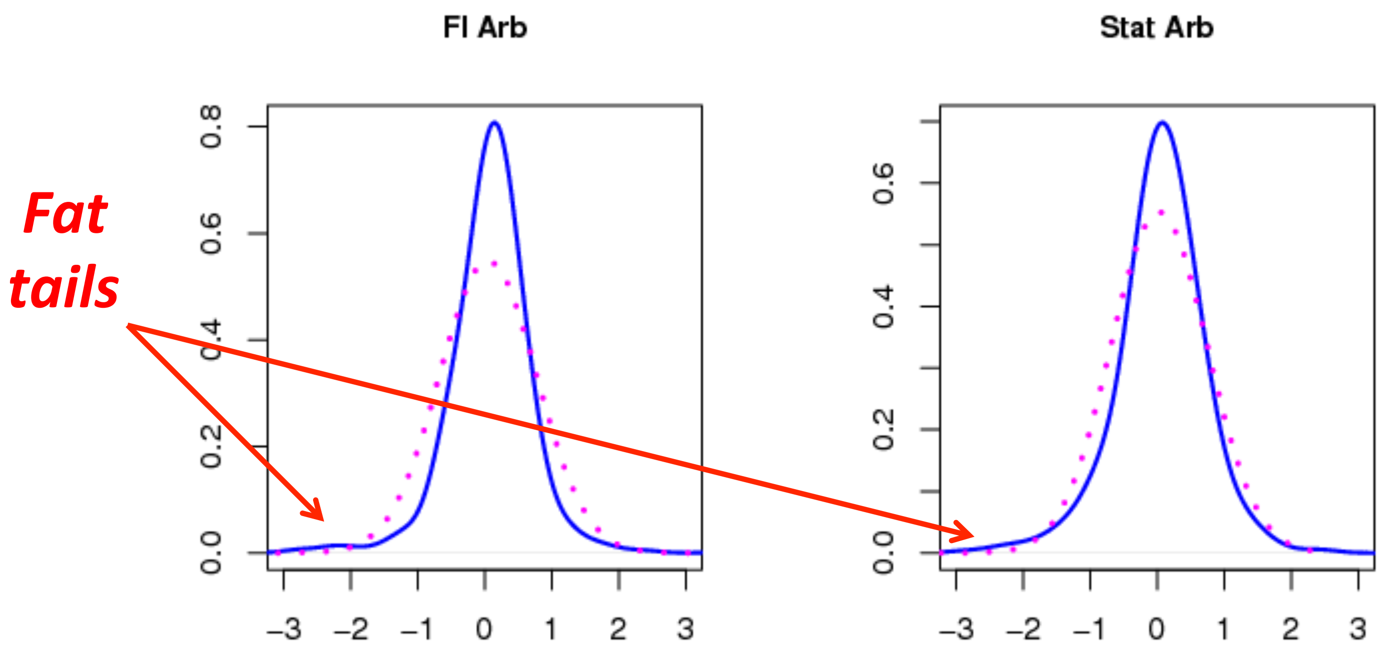
4. Empirical Applications – *hedge funds data*

Fixed Income versus Statistical Arbitrage: ***Marginal distribution***



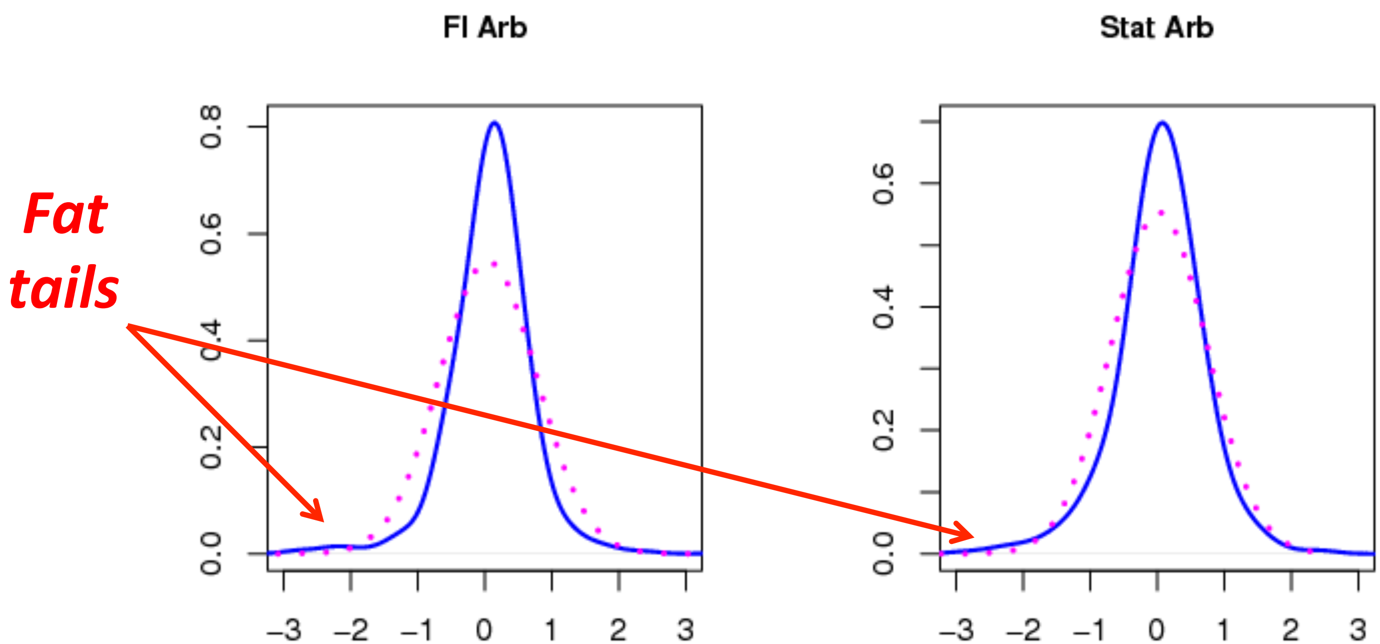
4. Empirical Applications – *hedge funds data*

Fixed Income versus Statistical Arbitrage: ***Marginal distribution***



4. Empirical Applications – *hedge funds data*

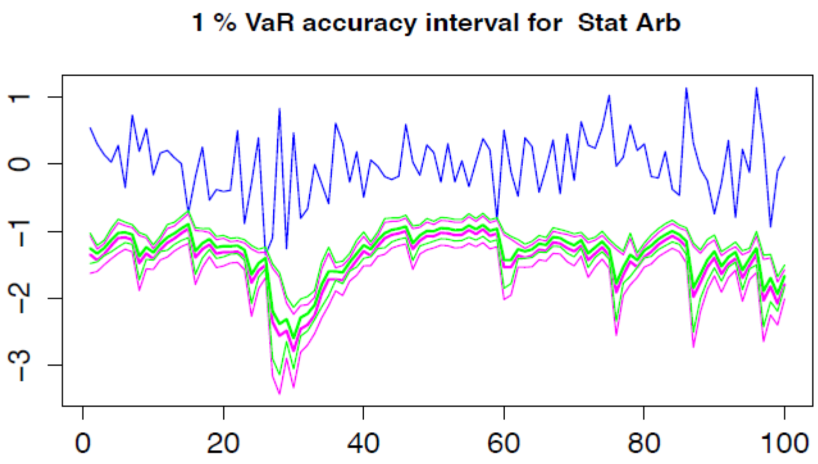
Fixed Income versus Statistical Arbitrage: ***Marginal distribution***

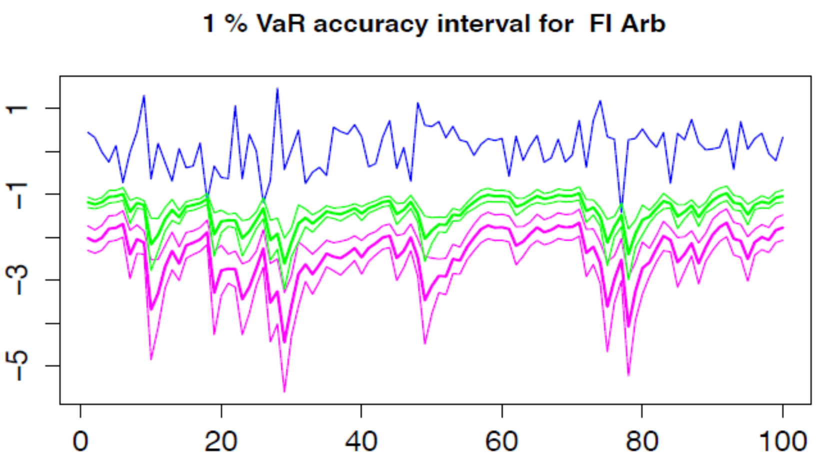


Volatility persistence or liquidity issue ?

4. Empirical Applications – *hedge funds data*

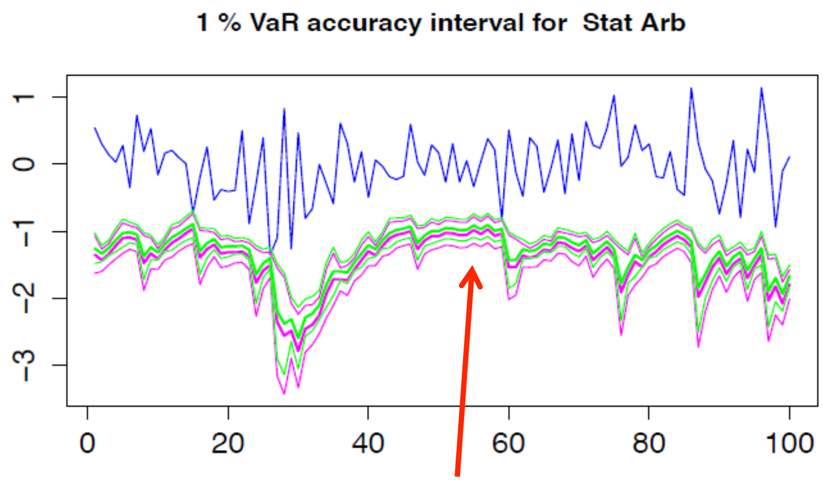
Fixed Income versus Statistical Arbitrage: ***Conditional distribution***

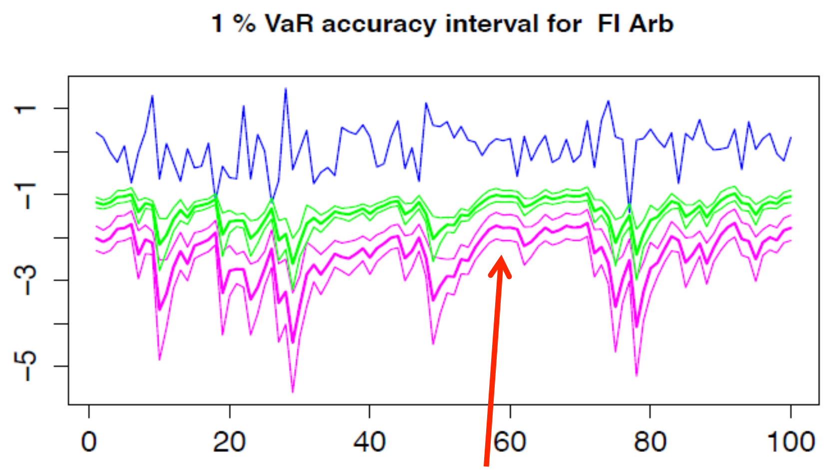




4. Empirical Applications – *hedge funds data*

Fixed Income versus Statistical Arbitrage: **Conditional distribution**



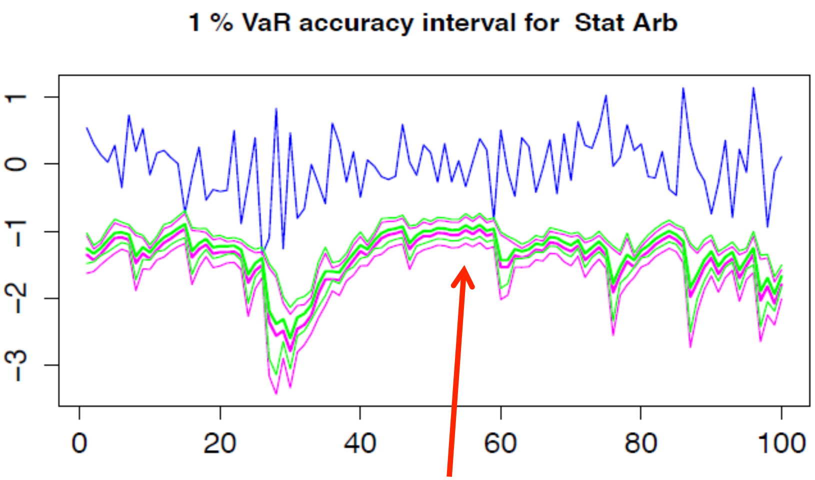


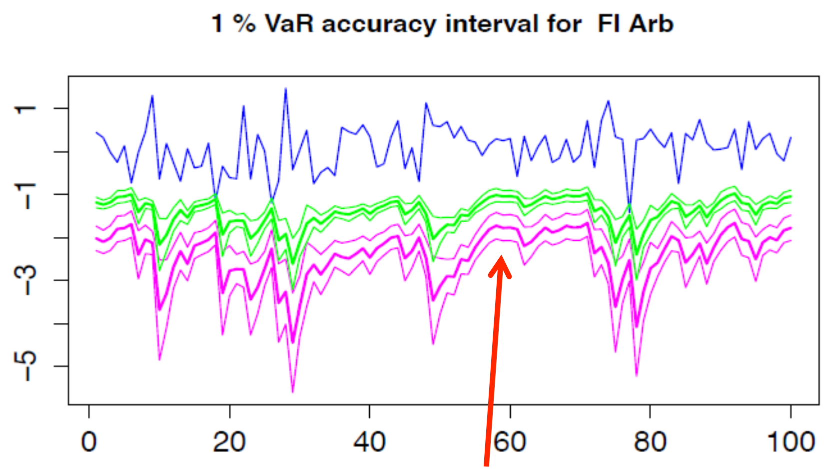
Normal tails: low liquidity risk

Fat tails: high liquidity risk

4. Empirical Applications – *hedge funds data*

Fixed Income versus Statistical Arbitrage: **Conditional distribution**





Normal tails: low liquidity risk

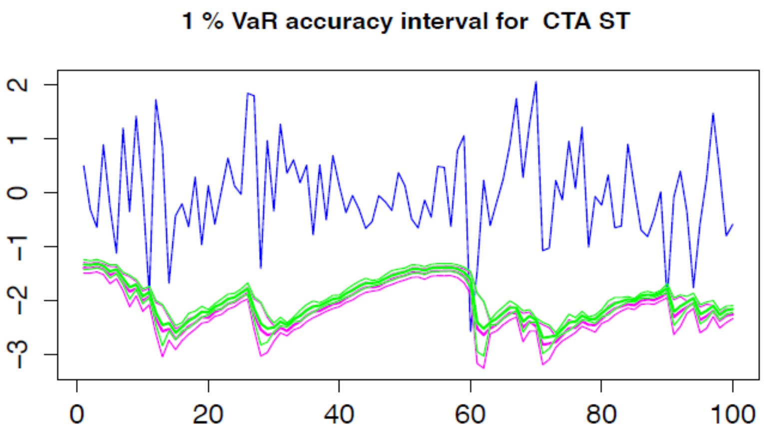
Fat tails: high liquidity risk

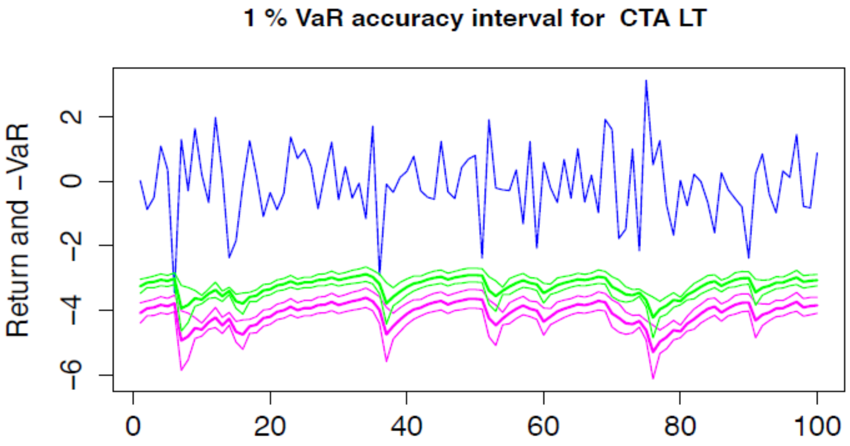
Stat Arb: *short term strategies on liquid assets (equities)*

FI Arb : *strategies on illiquid assets (bonds, credit derivatives, …)*

4. Empirical Applications – *hedge funds data*

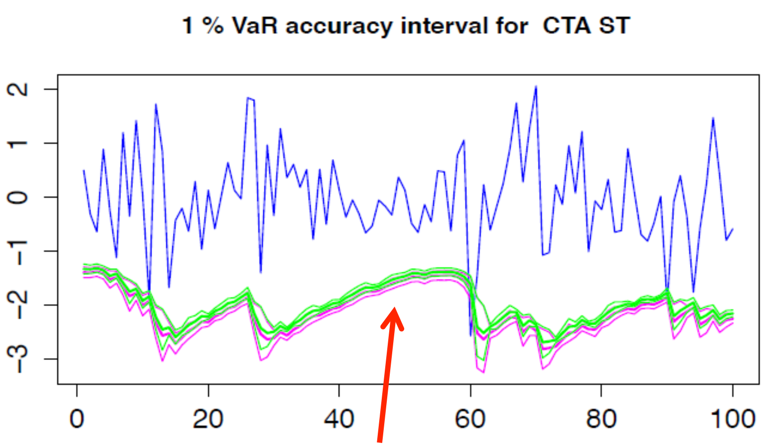
CTA Long Term versus CTA Short Term



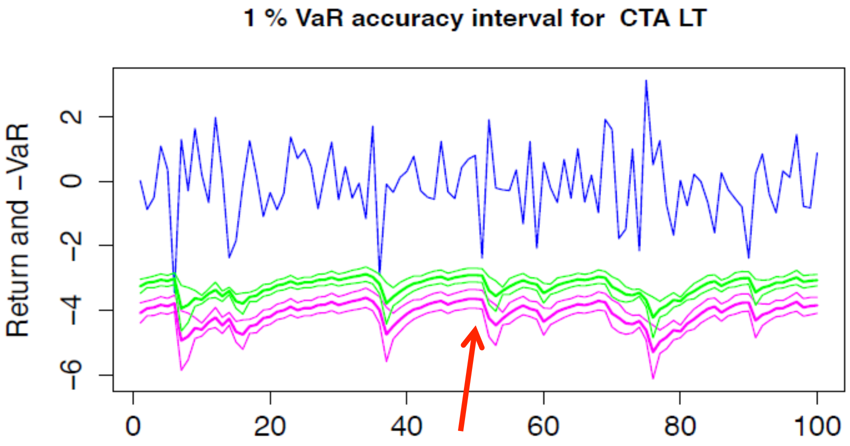


4. Empirical Applications – *hedge funds data*

CTA Long Term versus CTA Short Term



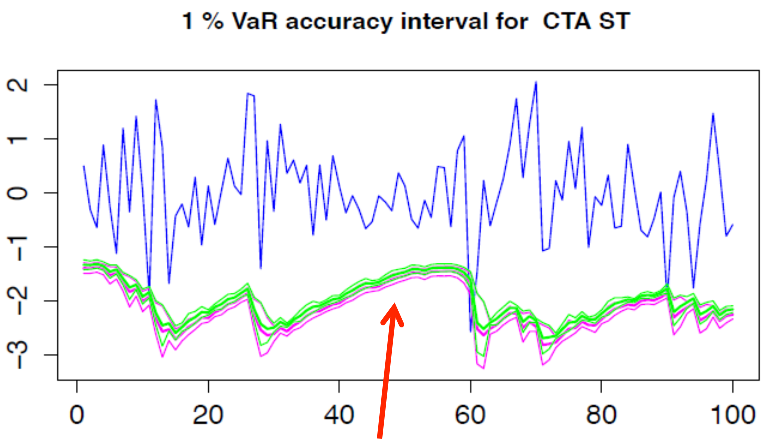
Normal tails: low liquidity risk

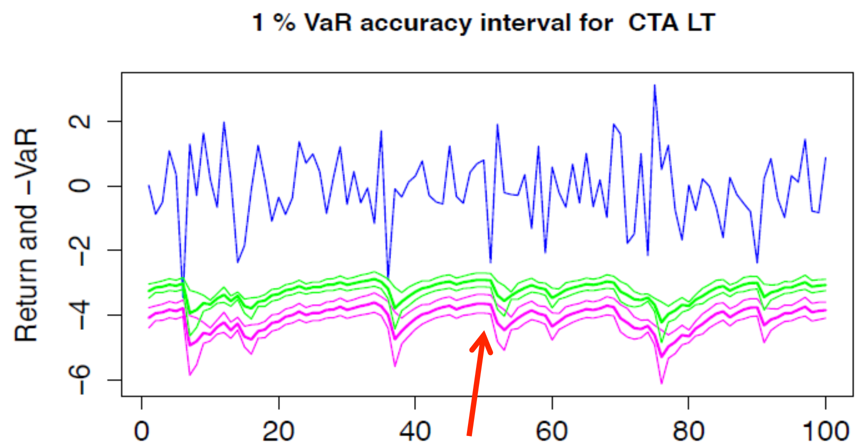


Fat tails: high liquidity risk

4. Empirical Applications – *hedge funds data*

CTA Long Term versus CTA Short Term





Normal tails: low liquidity risk

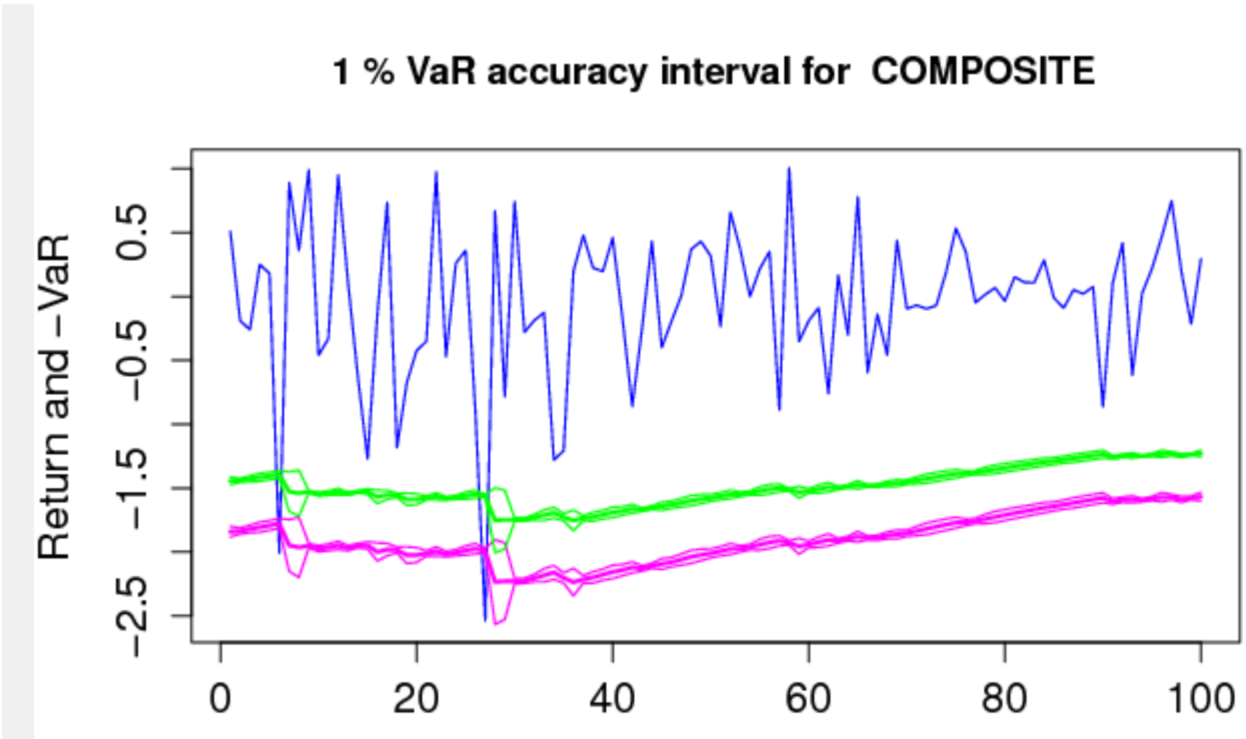
Fat tails: high liquidity risk

Same strategy … but on assets with different liquidity characteristics

CTA ST: *liquid assets* ***CTA LT:*** *possibly illiquid assets*

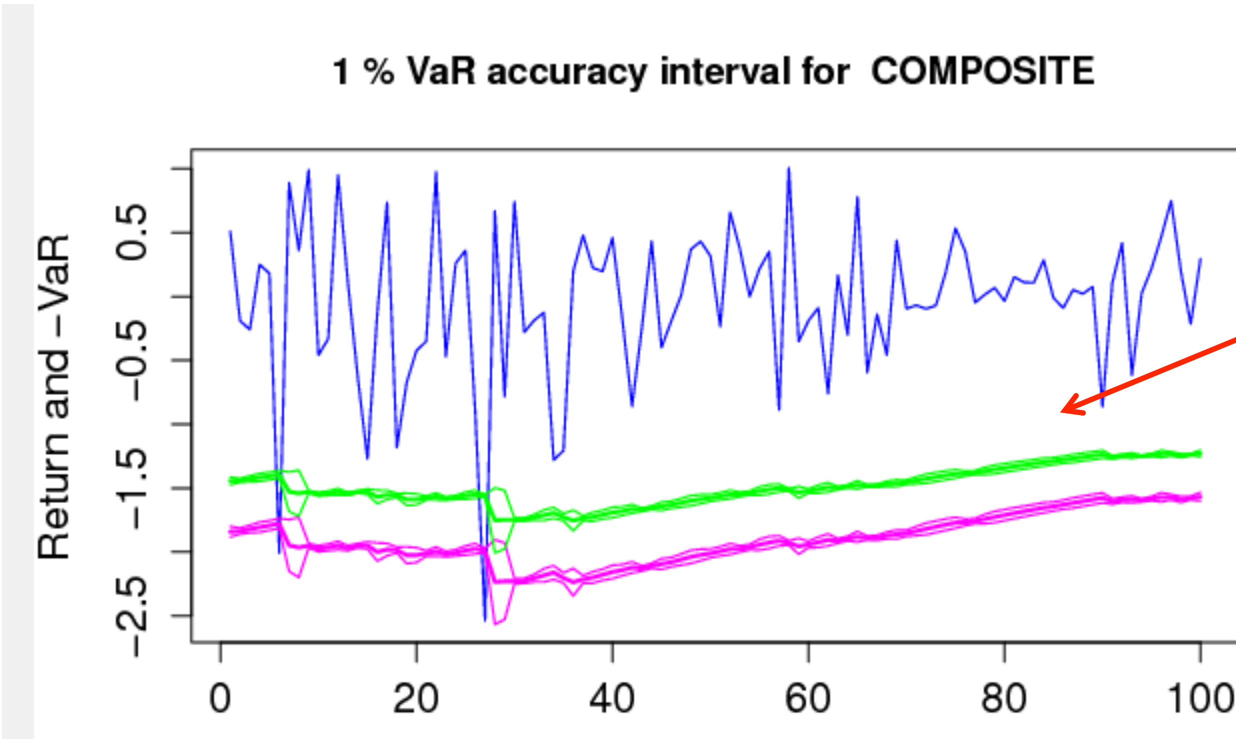
4. Empirical Applications – *hedge funds data*

Composite Index and Liquidity risk diversification



4. Empirical Applications – *hedge funds data*

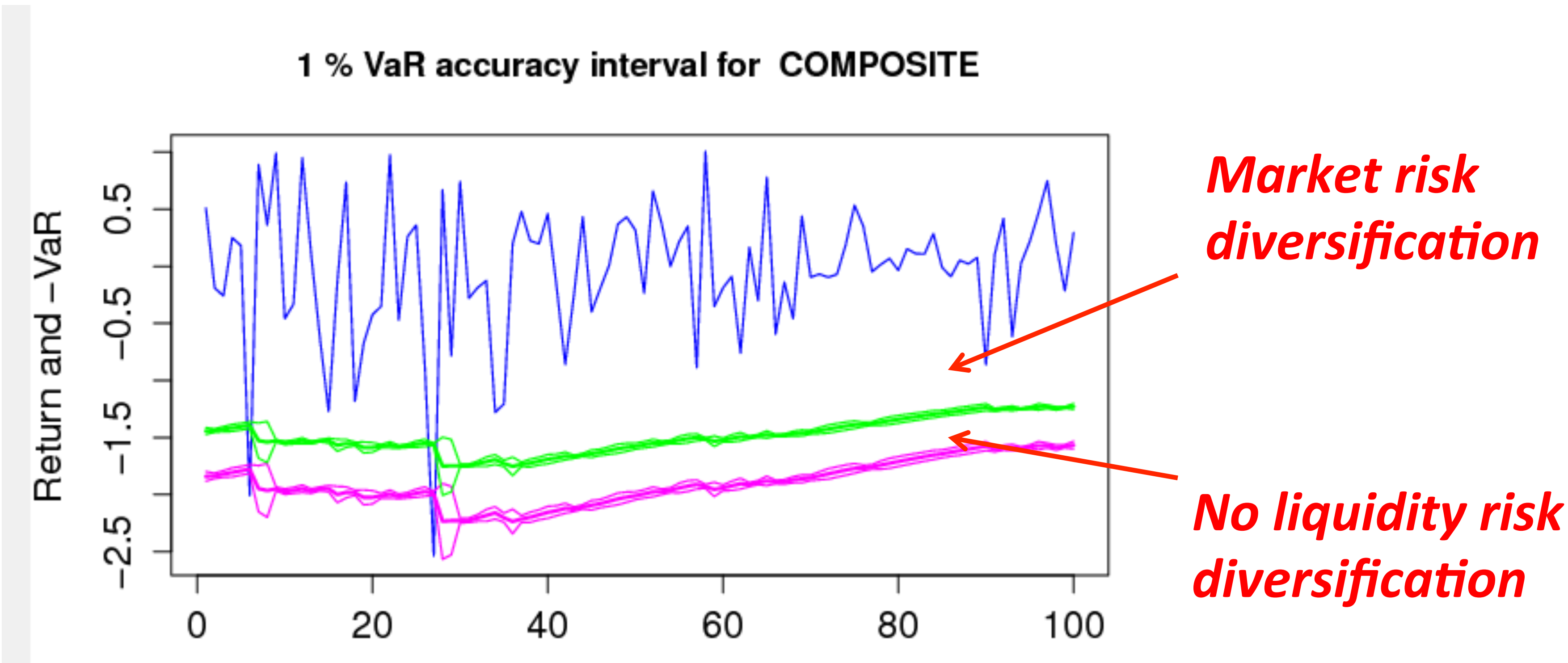
Composite Index and Liquidity risk diversification



Market risk diversification

4. Empirical Applications – *hedge funds data*

Composite Index and Liquidity risk diversification



Concluding Remarks

- *New liquidity risk-parameter*
- *Easy two-step estimation procedure*
- *Meaningfull resuls when applied to illiquid assets*

Ongoing research

- *Joint estimation of VaR related to different alphas*
- *Switching mechanism between different liquidity levels*